A rational tangle primer

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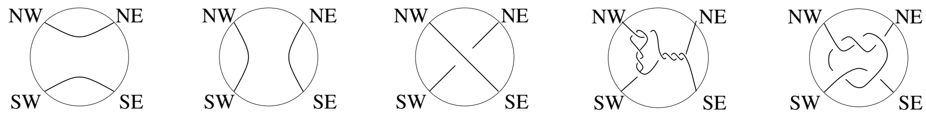
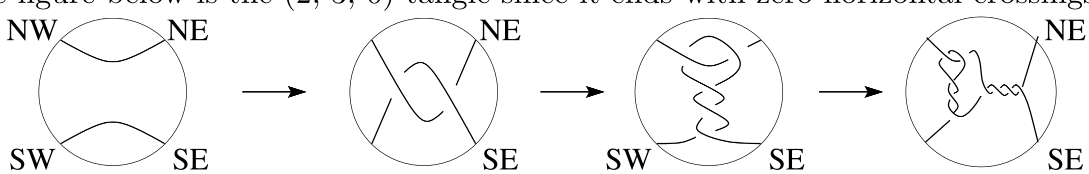
A 2-string tangle is a 3-dimensional ball containing two strings and a finite number of circles. Some examples:



(0) 0–tangle; (0,0) infinity tangle; (1) +1 tangle; (2, 3, 4) 30/7–tangle; a non–rational tangle

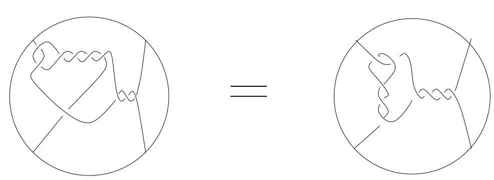
A tangle is rational if it can be obtained from the 0-tangle by alternating between rotating the NE and SE endpoints of the tangle and rotating the SW and SE endpoints of the tangle. For example, the tangle (2, 3, 4) is obtained from the 0-tangle by rotating NE and SE 2×180^o, followed by rotating SW and SE 3×180^o, and then rotating NE and SE 4×180^o. By convention, the tangle corresponding to $(c_1, ..., c_n)$ always ends with horizontal crossings. Thus the tangle with two horizontal crossings followed by three vertical crossings in the figure below is the (2, 3, 0) tangle since it ends with zero horizontal crossings.



(0) 0-tangle; (2) 2-tangle; (2, 3, 0) 2/7-tangle; (2, 3, 4) 30/7-tangle

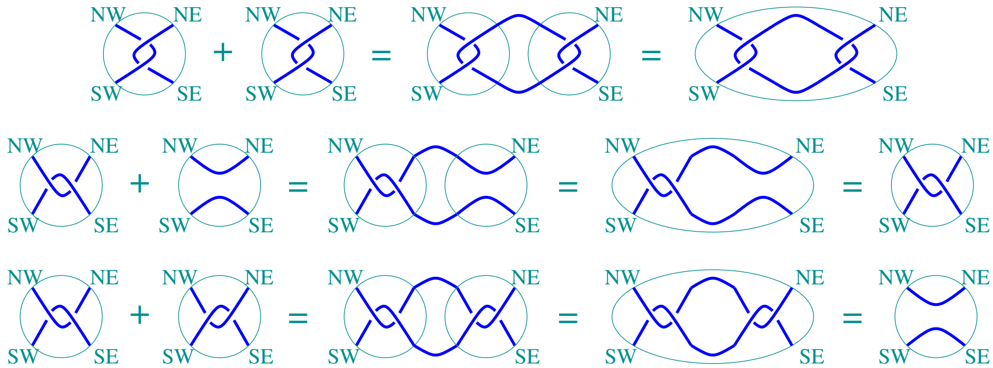
Rational tangles are uniquely identified by their corresponding continued fractions. For example, the tangle (2, 3, 4) is equivalent to the tangle (-1, -1, -4, 1, 3) since

$$4 + \frac{1}{3 + \frac{1}{2}} = \frac{30}{7} = 3 + \frac{1}{1 + \frac{1}{-4 + \frac{1}{-1 + \frac{1}{-1}}}}$$

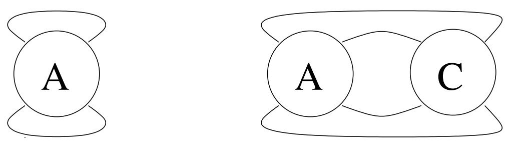


(-1, -1, -4, 1, 3) = (2, 3, 4)

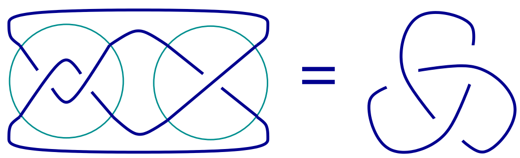
Two tangles can be added by connecting the NW endpoint of the first tangle to the NE endpoint of the second tangle and the SW endpoint of the first tangle to the SE endpoint of the second tangle. For example:



Knots and links can be formed by taking the numerator closure of a tangle or a sum of tangles:

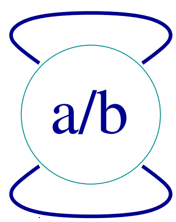


N(A) N(A+C)



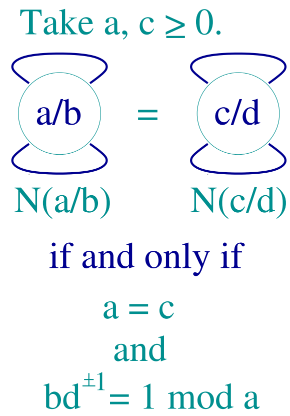
N(A + C) = trefoil knot

Definition: The numerator closure of a rational tangle is a rational knot or link (also called 4-plat or 2-bridge knot/link).



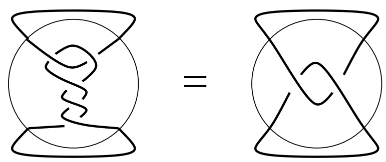
N(a/b)

Two rational knot/links are equivalent if and only if the following relationship holds.

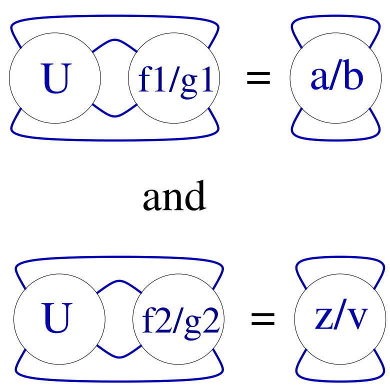


For example, $N((2,3,0)) = N(0 + \frac{1}{3+\frac{1}{2}}) = N(\frac{2}{7})$ and $N((2)) = N(\frac{2}{1}) = N((2))$

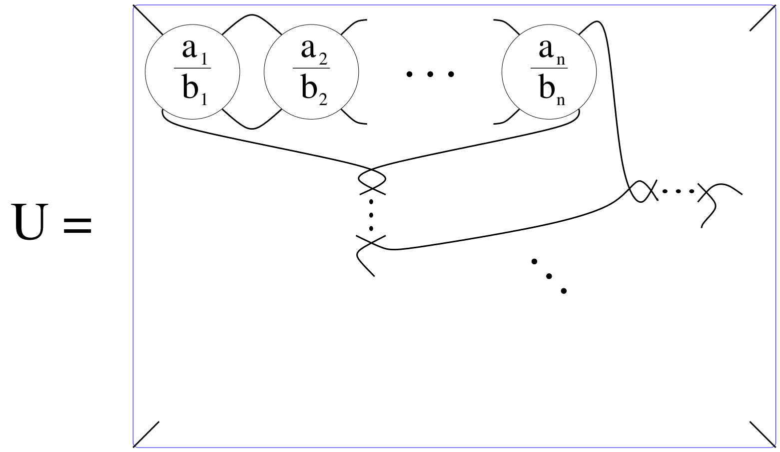
Thus, $N((2,3,0)) = N(\frac{2}{7}) = N(\frac{2}{1}) = N((2))$ since $7 = 1 + 2(3)$:



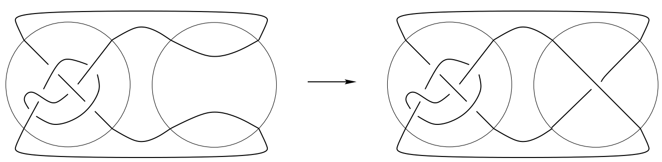
TopoICE-R solves the following system of tangle equations. The user inputs the knots $N(a/b)$ and $N(z/v)$ as well as the tangle f1/g1. TopoICE-R solves for the tangles U and f2/g2.



TopoICE-R assumes U is of the following form.



Since $N(a/b)$ and $N(z/v)$ are rational knots, $n = 1, 2$. If |f1g2 - f2g1| > 1, then U must be of this form. Hence TopoICE-R finds all solutions in this case. When |f1g2 - f2g1| = 1, TopoICE-R may not find all solutions since U may not be of this form. For example if the rational knot $N(11/4)$ and the rational link $N(2/1)$ is entered in TopoICE-R and if f1/g1 is chosen to be 0/1, then TopoICE-R misses the following solution where f2/g2 = 1/1 (observe f1g2 - f2g1 = (0)(1) - (1)(1) = -1) and U is not isotopic to a sum of rational tangles.



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